The Migdal-Kadanoff approximation: optimisation, generalisation, Ising model in external fields and the Migdal-Kadanoff hierarchies



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# The Migdal–Kadanoff approximation: optimisation, generalisation, Ising model in external fields and the Migdal–Kadanoff hierarchies

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**Abstract.** The simplest Migdal–Kadanoff approximation is optimised on hypercubic lattices. Generalisation is made to all regular lattices leading to recursion relations continuous in lattice coordination. One variety of these relations shows universality whilst another variety shows duality on planar lattices. The Ising model in external fields, both normal and staggered, is investigated. Phase diagrams and exponents are found under a complete treatment of the ambiguity in the handling of external fields by bond-moving. Finally the underlying exactly solved family of hierarchical lattices is considered. It is shown that these lattices exhibit a complete breakdown in the concept of universality.

## 1. Introduction

The simple renormalisation approximations due to Migdal (1975) and Kadanoff (1976) have received much attention and application in the literature. Many authors have sought to improve the basic scheme (Emery and Swendsen 1977, Nicoll 1979, Maritan 1980, Martinelli and Parisi 1981, Lipowsky 1982). Many models of contemporary interest have been explored with the approximations (José *et al* 1977, Domany and Riedel 1979, Andelman and Berker 1981, and many others). It has been recognised (Berker and Ostlund 1979) that recursion relations found on regular lattices under the approximations constitute exact solutions on types of fractal lattice termed *hierarchical lattices* by Kaufman and Griffiths (1981).

The essential approximation within the schemes is the distortion of lattices, at each renormalisation step, by moving bonds such that tractable decimations may be carried out. In the simplest schemes (Migdal 1975, Kadanoff 1976, Emery and Swendsen 1977), the distorted lattice is an isomorphic superlattice with strings of spins decorating its edges (see figure 2). Decimating the strings on the decorations leads to recursion relations for the nearest neighbour couplings. This paper is concerned with these simple *string-based* schemes. Section 2 describes the orginal versions of the approximations and discusses the well known features. Section 3 shows how the bond-moving approximation can be interpreted in a liberal fashion which allows free parameters to be introduced and an optimal approximation to be found on the hypercubic lattices. In § 4 recursion relations applicable to all regular lattices are found. These relations are continuous in both dimension and lattice coordination. One variety shows universality in exponents whilst another shows duality on planar lattices. The approximations are well known to be ambiguous in their treatment of

 

external fields. In § 5 this ambiguity is examined for the case of external fields on the Ising model. Understanding the recursion relations as exact solutions to hierarchical lattices is essential and § 6 contains comments and results on this. Throughout the work results on the Ising model are used to discuss and establish points.

## 2. The original approximations

Firstly some notation is introduced. Instead of bonds, a decorated lattice (Syozi 1972) is one which has more complicated structures, decorations, between the vertices. A general such decoration is $M$ strings of length $A$ with $B$ multiple bonds between each pair of vertices. Figure 1 illustrates this. (A further generalisation, considered below, consists of decorations with strings having varying $B$ and $M$ factors.)

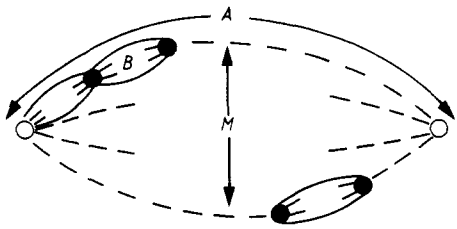


**Figure 1.** A general decoration, $M$ strings of length $A$ with $B$ bonds per coupling. Lattice sites are shown as open circles.

Kadanoff (1975) introduced the bond-moving approximation. If bonds are moved between equivalent sites on a lattice then the free energy on the distorted lattice is a lower bound to that of the original lattice. Using this Kadanoff (1976) was able to derive recursion relations which involved that found earlier by Migdal (1975) and that found later, using another bond-moving construction, by Emery and Swendsen (1977, to be referred to as ES). In all these approximations a decorated isomorphic superlattice, of lattice spacing $b$ times the original spacing, is formed. The decorations constructed by each are

| | | | | |
|---|---|---|---|---|
| Kadanoff: | $A_i = b$ | $B_i = b^{(d-i)}$ | $M_i = b^{(i-1)}$ | (1*a*) |
| Migdal: | $A = b$ | $B = 1$ | $M = b^{(d-1)}$ | (1*b*) |
| ES: | $A = b$ | $B = b^{(d-1)}$ | $M = 1.$ | (1*c*) |

Figure 2 illustrates on the square lattice the Migdal and ES constructions for $b = 2$. The coefficients in (1*a*) and (1*b*) are applicable to hypercubic lattices, whereas the ES construction may be carried out on any regular lattice by simply choosing a supercell and moving all interior bonds onto the edge strings. The Kadanoff construction introduces anisotropic couplings and the $i$th coefficients hold for the $i$th lattice direction. However, recursion relations for each direction are independent of couplings in other directions. This leads to a pathological treatment of the anisotropic model. The pathologies are both global, as noted by Kadanoff (1976), and local around the fixed point (the fixed point is isolated with $d$ equal and relevant scaling eigenvalues). The ES construction, though applicable to all regular lattices, depends only on their

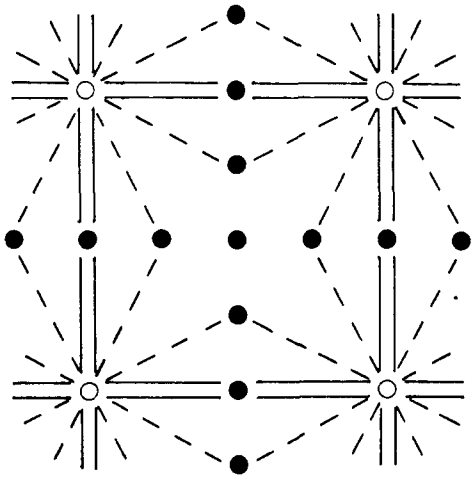

**Figure 2.** The $b = 2$ Migdal (broken lines) and ES (full lines) decorated superlattices on the square lattice.

dimension: a universality expected for critical exponents but not critical couplings or thermodynamic functions.

As noted by Kadanoff (1976) the constructions do possess duality features shown by planar lattices. In particular a planar lattice with ES type decorations is dual to one with Migdal type decorations. Figure 3 shows the necessary construction for the case $b = 2$ on the square lattice. It is required that each vertex of one lattice is surrounded by an elementary cycle on the other and *vice versa*. In general, lattices

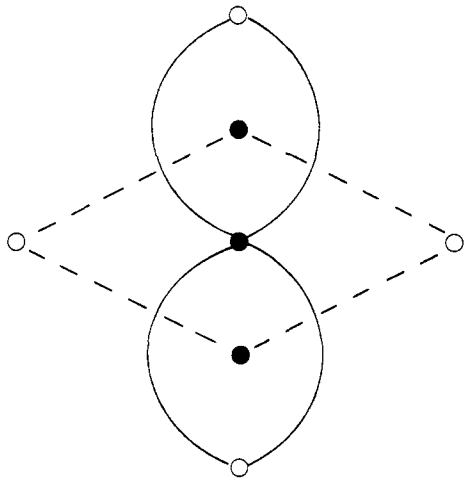

**Figure 3.** The duality relationship between the Migdal (broken lines) and ES (full curves) decorations (case $d = 2$ and $b = 2$).

with decorations ($A = A$, $B = B$, $M = 1$) are dual to dual lattices with decorations ($A' = B$, $B' = 1$, $M' = A$). Fixed points for dual relations are related by duality transformations.

The merits and drawbacks of the approximations are well known (Kadanoff 1976). On the merit side are: the continuity of recursion relations in dimension and the ease with which many models may be examined; the availability of differential transformations; the duality properties outlined above; a presumed exactness to first order in $\varepsilon$,

where $d = d_{\text{lcd}} + \varepsilon$ ($d_{\text{lcd}}$ = lower critical dimensionality) (Migdal 1975). Against this are weighed: poor thermal eigenvalues; negative specific heat exponents for all $d$; the dependence of results on $b$; a worsening of results with increasing dimension and incorrect high-dimensional limits of exponents. Furthermore, cases of incorrect phase diagrams, not only at high dimension but also in planar systems, have been noted (see, for example, the anomalies discussed by Andelman and Berker (1981), Kaufman *et al* (1981) and Domany and Riedel (1979)).

The approximations are of a decimation transformation which is not expected to be a proper renormalisation (Wilson and Bell 1974, Sneddon and Barber 1977). Improvement schemes, although successful in giving better exponents, should bear this in mind. However, it is clear that including further neighbour and multiple spin couplings directly under bond-moving constructions (Lipowsky 1982) or within perturbation schemes (Martinelli and Parisi 1981) does lead to better estimates of exponents.

## 3. Generalised bond-moving and optimal approximation on hypercubics

The bond-moving approximation can be interpreted in a more liberal fashion than hitherto in the literature. Following the usual prescription (Kadanoff 1976) bond-moves are specified by the addition of a perturbation $V$ to the Hamiltonian $H$. The bond-move condition is $\langle V \rangle_H = 0$, which guarantees that free energy estimates are bounded from below, and is satisfied if bonds are moved between equivalent sites. Consider $m$ successive bond-moves, $V = V_1 + \ldots + V_i + \ldots + V_m$. At the $i$th move one may specify that either

$$\langle V_i \rangle_{(H+V_1+\ldots+V_{i-1})} = 0 \tag{2a}$$

or just

$$\langle V_i \rangle_H = 0. \tag{2b}$$

Under (2$a$) the $i$th moved bonds may only be moved between sites equivalent on the $(i-1)$th distorted lattice. However, under (2$b$) sites need only be equivalent on the original lattice. The free energy bound is maintained on the full bond-move under both (2$a$) and (2$b$), yet (2$b$) is a much weaker condition on possible moves than (2$a$); furthermore, there is no reason why (2$b$) should lead to worse estimates. (Restricting moves to condition (2$a$) leads to the necessity of anisotropic decorations under Kadanoff's construction.)

Using the weaker condition it is possible to make a wide variety of decorated superlattices. An example on the square lattice is shown in figure 4. It is clear that on the hypercubics all such decorations must be formed with strings of length $A = b$. The variation available is in the $B$ and $M$ factors. Rather than bother about explicit construction, consider $r$ types of string specified by $r$ sets of coefficients $B_i$, $M_i$. One may choose to leave some spins disconnected as in the ES construction or even to leave disconnected finite segments of lattice as in the examples discussed by Swendson and Zia (1979). The failure of the latter scheme is noted and in this paper all spins and bonds are included in the decorated superlattice. On hypercubic

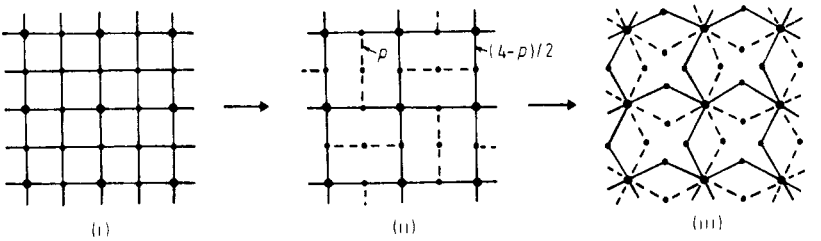


**Figure 4.** Bond-moving construction involving a free parameter $p$ on the square lattice with $b = 2$. From (i) to (ii) two bonds in each supercell are moved to form the broken couplings with $B = p$ bonds and full couplings with $B = (4-p)/2$ bonds. From (ii) to (iii) the strings with broken couplings are moved onto the cell edges to leave them with $M = \frac{1}{2}$ strings with $B = p$, and $M = 1$ strings with $B = (4-p)/2$. Note that the move from (ii) to (iii) is of the type $(2b)$ discussed in the text.

lattices this conservation of spins and bonds leads to the following constraints on the coefficients of the decorations:

$$\text{conserving spins:} \qquad (A-1)\sum_i M_i = (b^d - 1)/d \tag{3a}$$

$$\text{conserving bonds:} \qquad A\sum_i M_i B_i = b^d. \tag{3a}$$

As stated earlier $A = b$ and the decorated superlattice is taken to be isotropic. Assuming that the free energy bound is obeyed by all constructions, the $M_i$ and $B_i$ may be treated as variational parameters (under $(3a)$ and $(3b)$) and chosen by optimising the free energy estimate. This is carried out in appendix 1 and the rather obvious result is that the optimal decoration is the homogeneous one, all strings being the same. From $(3a)$ and $(3b)$ one finds this to be

$$A = b \qquad B = d(b^d - b^{d-1})/(b^d - 1) \qquad M = (b^d - 1)/[d(b-1)]. \tag{4}$$

The conditions $(3a)$ and $(3b)$ imply that free energy estimates are exact at high and low temperatures. As discussed in appendix 1, on these simple string-based approximations optimising the free energy estimate corresponds to optimisation of the thermal exponents (which should be contrasted with more sophisticated approximations such as those of Lipowsky (1982) and Kadanoff *et al* (1976)).

The author has examined the coefficients (4) on the Ising model

$$-\beta H = \sum_{ij} JS_iS_j \qquad (S_i = \pm 1)$$

where $\beta = 1/k_B T$, $k_B$ is the Boltzmann constant and $T$ the temperature. Recursion relations found for the general case of coefficients $ABM$ are

$$\tanh(J'/M) = \tanh^A(BJ) \tag{5}$$

and

$$K = \tfrac{1}{2}M \ln[2^{2A-2}(\cosh^{2A}(BJ) - \sinh^{2A}(BJ))] \tag{6}$$

where $J'$ is the renormalised coupling and $K$ is the free energy contribution per edge of the new lattice. The usual analysis is pursued (Niemeyer and van Leeuwen 1973) and the coefficients (4) are compared with (1*a*), (1*b*) and (1*c*). Figures 5 and 6 respectively show the dependence on $b$ of the single unstable fixed point, $J^*$, and the

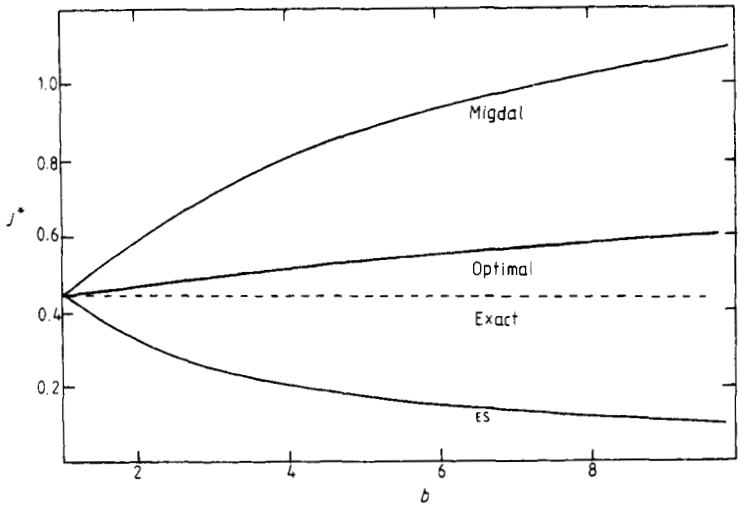


**Figure 5.** The unstable fixed point $J^*$ against $b$ for the case $d = 2$. The Migdal (1*b*), ES (1*c*) and optimal (4) coefficients are compared.

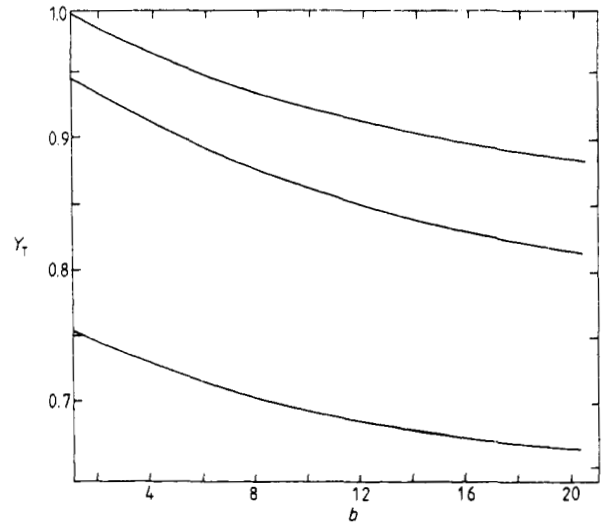


**Figure 6.** The thermal scaling dimension $Y_T$ against $b$ for $d = 2$, 3 and 4. As $b \to 1$, $Y_T = 0.7536$, 0.9482 and 0.9899, respectively.

thermal scaling dimension, $Y_T$, defined at $J^*$. The estimate for $J^*$ given by (4) is superior to that of (1*b*) and (1*c*) in the case shown (the square lattice). However, all coefficients give the same $Y_T$ curves. As indicated by the graphs, all coefficients lead

to the same results and differential transformation as $b \to 1$. Free energy estimates may be computed using (6) (Nauenberg and Nienhuis 1974). Figure 7 compares the exact solution on the square lattice (Onsager 1944) and the estimates from (4), (1*a*),

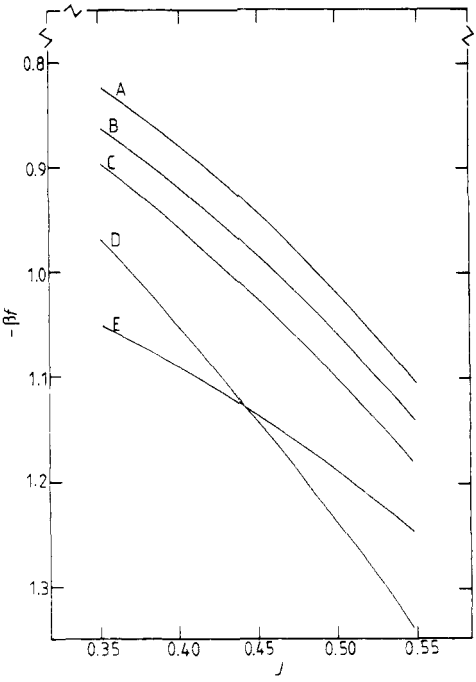


**Figure 7.** Free energy estimates on the square lattice for the case $b = 2$. A, exact result (Onsager 1944); B, optimal coefficients (4); C, Kadanoff (1*a*); D, ES (1*c*); E, Migdal (1*b*).

(1*b*) and (1*c*). The case $b = 2$ is shown for a region around the critical coupling. The optimisation of (4) is evident and shown over the full coupling range. Estimates are poorest in the critical region.

## 4. Generalisation to all regular lattices

Under the weaker bond-moving condition (2*b*) constructions may be carried out on any regular lattice. An example on the triangular lattice is shown in figure 8. On most lattices strings of length $A = b$ need to be formed; however, on the hexagonal lattice, for example, strings of length $A = b^{4/3}$ are necessary. Rather than bond-moving a more *ad hoc* but instructive route to generalisation is undertaken. Following the results of the last section, homogeneous, isotropic decorations are assumed. On a regular lattice of dimension $d$ and coordination $z$ the conditions (3*a*) and (3*b*) generalise to

$$M(A-1) = (2/z)(b^d - 1) \tag{7a}$$

$$MBA = b^d. \tag{7b}$$

The coefficient $A$ will be left free and generalisation sought by examining the dependence on $A$ of Ising model results.

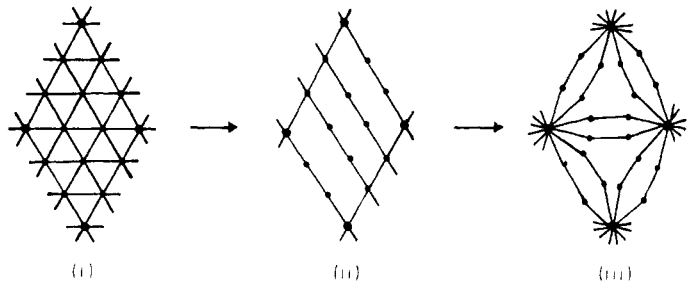

**Figure 8.** Construction on the triangular lattice with $b = 3$. From (i) to (ii) strings with $B = \frac{9}{4}$ are formed. From (ii) to (iii) each supercell edge is given $M = \frac{4}{3}$ of these strings.

The differential transformation is examined initially. It is convenient to set $A = b^a$. Letting $b = \exp(\delta)$, expanding (5) in $\delta$ and applying (7$a$) and (7$b$) leads to

$$dJ'/d\delta = Jd[1-(a/d)]+(2d/z)X(zaJ/2d) \tag{8}$$

where $X(x) = \tanh x \ln(\tanh x)/(1-\tanh^2 x)$. Furthermore, $Y_T$ is found as $\delta \to 0$ to be given by

$$Y_T = d[1+(2a/d)X(zaJ^*/2d)/\tanh(zaJ^*/d)]. \tag{9}$$

For $0 < (a/d) < 1$, a single unstable fixed point is found. Figures 9 and 10 respectively show the variation of $J^*$ and $(Y_T/d)$ with $(a/d)$ and $z$. The $(Y_T/d)$ curve is universal (independent) with respect to $z$.

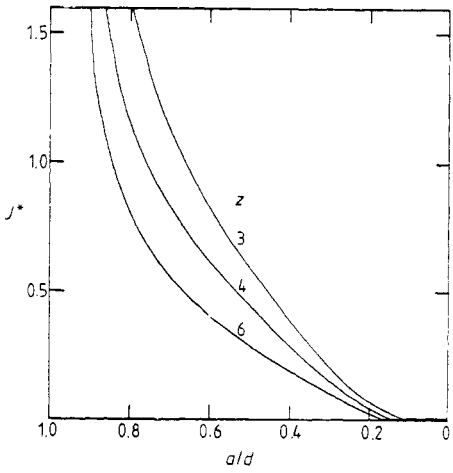


**Figure 9.** The unstable fixed point of (8), $J^*$, against $a/d$ with $z = 3$, 4 and 6. For $a/d < 0$ and $a/d > 1$ no unstable fixed point exists. As $a/d \to 0$, $J^* \to 0$; as $a/d \to 1$, $J^* \to 1/z(1-a/d)$.

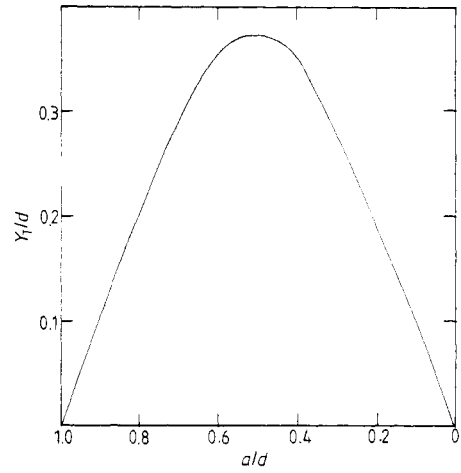


**Figure 10.** $Y_T/d$ against $a/d$ for (8) and (9). The curve has a maximum at $a/d = 0.5$ of $Y_T/d = 0.3768$ and is independent (universal) of $z$.

The coefficients (1$a$), (1$b$), (1$c$) and (4) all have a common differential transformation

$$dJ'/d\delta = J(d-1)+X(J). \tag{10}$$

Now $a$ is chosen such that (8) reduces to (10) on the square lattice; the following are appropriate

$$\begin{array}{lll} \text{(i)} & a = 2d/z & (A = b^{2d/z}) \\ \text{(ii)} & a = d/2 & (A = b^{d/2}) \\ \text{(iii)} & a = 1 & (A = b). \end{array} \tag{11}$$

For (i) and (iii) (8) reduces to (10) on the hypercubics ($2d = z$). As mentioned earlier, on most lattices bond-moving constructions lead to $A = b$ which is choice (iii) above. In table 1 results for each choice are compared with each other and the known results.

Referring to table 1, for (i) and (ii) $J^*$ depends only on $z$. Only (iii) shows a dependence of $J^*$ on $d$ and $z$ which is qualitatively correct. Universality in $Y_T$ is shown by (ii) and (iii) but not by (i). Choice (ii) gives a maximal $Y_T$ for all $d$. Coefficients

**Table 1.** Comparison of results from generalisations with known results: $d = 2$, exact (Onsager 1944), $d = 3$, critical couplings (Domb 1974) and exponent (Le Guillou and Zinn Justin 1980, see also Roskies 1981) and $d = 4$, critical coupling (Fisher and Gaunt 1964) and mean field exponent.

| | | Known | | (i) $a = 2d/z$ | | (ii) $a = d/2$ | | (iii) $a = 1$ | |
|---|---|---|---|---|---|---|---|---|---|
| Lattice | $(d, z)$ | $J^*$ | $Y_T$ | $J^*$ | $Y_T$ | $J^*$ | $Y_T$ | $J^*$ | $Y_T$ |
| Hexagonal | (2, 3) | 0.6584 | 1 | 0.9871 | 0.6321 | 0.5876 | 0.7536 | 0.5876 | 0.7536 |
| Square | (2, 4) | 0.4407 | 1 | 0.4407 | 0.7536 | 0.4407 | 0.7536 | 0.4407 | 0.7536 |
| Triangular | (2, 6) | 0.2746 | 1 | 0.1398 | 0.6321 | 0.2938 | 0.7536 | 0.2938 | 0.7536 |
| Diamond | (3, 4) | 0.3698 | 1.587 | 0.4407 | 1.1304 | 0.4407 | 1.1304 | 0.2906 | 0.9482 |
| Cubic | (3, 6) | 0.1021 | 1.587 | 0.1398 | 0.9482 | 0.2938 | 1.1304 | 0.1398 | 0.9482 |
| BCC | (3, 8) | 0.1574 | 1.587 | 0.0501 | 0.7425 | 0.2203 | 1.1304 | 0.1048 | 0.9482 |
| FCC | (3, 12) | 0.1021 | 1.587 | 0.0067 | 0.4998 | 0.1469 | 1.1304 | 0.0699 | 0.9482 |
| Hypercubic | (4, 8) | 0.1498 | 2 | 0.0501 | 0.9899 | 0.2203 | 1.5072 | 0.0501 | 0.9899 |

for general $b$ can be found from (i), (ii) and (iii) with (7$a$) and (7$b$). For all three choices $J^*$ increases and $Y_T$ decreases with increasing $b$. For (ii) and (iii) values of $Y_T$ remain universal for all $z$. Varying $A$ reveals that $Y_T$ is a maximum for (ii), $A = b^{d/2}$, for all $b$. As $d \to 1$ the approximation should become exact; this is not shown by (ii). It is noted that (i) is found on the hexagonal lattice by bond-moving (as mentioned earlier) and hence that bond-moving breaks universality.

The preference of the author is to consider (iii) as the natural extension of (10) and (4) to the regular lattices. However, a well known feature of (10) is that it shows the self-duality of the square lattice. The choice (iii) does not extend this feature whereas the choice (i) does.

To discuss the duality of (i) some notation is introduced. Decimation of the string decorations can best be carried out by using transfer matrices. Assuming discrete spin models, denote their coupling by a matrix $I$ and define the transfer matrix $T$ via $T_{ij} = \exp(I_{ij})$. Extending a notation due to Nicoll (1979) the transformation for the general decoration of figure 1 is written

$$\exp(K)T'_{ij} = [T^A(T^B_{pq})]^M_{ij} \tag{12}$$

where $K$ is a constant and on the right-hand side the elements of $T$ are raised to the power $B$, $T$ is raised to the power $A$ and then the $ij$th element of the resultant is raised to the power $M$ (multiple bonds and strings combine by direct product). Now setting the coefficients via (i), (7$a$) and (7$b$) and expanding as before one finds

$$\mathrm{d}T_{ij}/\mathrm{d}\delta = d(1-2/z)T_{ij}\ln(T_{ij}) + (2d/z)[T\ln(T)]_{ij} \tag{13}$$

where multiplicative constants are omitted above and below. Wu and Wang (1976) formulated the duality transform on planar lattices for the general family of models possessing cyclic transfer matrices. Under this transform the elements of the dual transfer matrix are the eigenvalues of the original matrix and *vice versa*. Denoting the distinct elements of $T$ by $t_k$, the eigenvalues (of which there are the same number as the $t$'s) by $r_l$ and the $j$th element of the $l$th eigenvector by $f_{jl}$, equation (13) may be written

$$\frac{\mathrm{d}t_k}{\mathrm{d}\delta} = d(1-2/z)t_k\ln(t_k) + (2d/z)\sum_l r_l\ln(r_l)f_{kl}. \tag{14}$$

Now using

$$\frac{\mathrm{d}r_p}{\mathrm{d}\delta} = \sum_k f_{kp}\frac{\mathrm{d}t_k}{\mathrm{d}\delta} \tag{15}$$

one finds

$$\frac{\mathrm{d}r_p}{\mathrm{d}\delta} = d(1-2/z)\sum_j t_j\ln(t_j)f_{pj} + (2d/z)r_p\ln(r_p). \tag{16}$$

Equations (14) and (16) are respectively the transformation of the original model on the original lattice and that of the dual model on the dual lattice. From these it is seen that the coordinations of original lattice and dual lattice are related by $z' = 2(1-2/z)^{-1}$, which is the correct topological relationship between planar dual lattices.

## 5. Treatment of the Ising model with external fields

As pointed out by José *et al* (1977), the treatment of external fields under bond-moving distortions is ambiguous. Any Hamiltonian can be trivially rearranged such that fields may be written as site terms or as part of the bonds. Such rearrangements do not commute with subsequent bond-moving. Results found from recursion relations differ both qualitatively and quantitatively between different arrangements.

The Ising model is considered with both an external field $h$ conjugate to the ferromagnetic order parameter and a staggered field $f$ conjugate to the antiferromagnetic order parameter. A parameter $q$ is introduced and a fraction $q$ of each site's field is included with its bonds. The Ising Hamiltonian is written

$$-\beta H = (1-q)\Big(h\sum_i S_i + f\sum_{j_1} S_{j_1} - f\sum_{j_2} S_{j_2}\Big)$$
$$+ J\sum_{ij} S_iS_j + \frac{2qh}{z}\sum_{ij}(S_i+S_j) + \frac{2qf}{z}\sum_{ij}(S_i-S_j) \tag{17}$$

where two sublatttices are assumed and $j_1$ and $j_2$ respectively denote spins on sublattice (1) and sublattice (2). The first line consists of the site terms and the second line consists of the bond terms which are moved.

José *et al* (1977) in their investigation of the planar model used $q = 1$ for symmetry-breaking fields. They made this choice on the basis of the agreement of results with those from a spin-wave theory. Emery and Swendson (1977) suggested that $q = 0$ was appropriate for crystal fields. They suggested that using $q$ other than zero leads to an incorrect treatment of external fields. However, decimation is not expected to be a proper renormalisation transformation (Wilson and Bell 1974) and other schemes such as majority rule block-spins (Niemeyer and van Leeuwen 1973) do not show fixed fields at zero coupling (as may easily be shown). Phase diagrams differ between $q = 1$ and $q = 0$. For the spin-one model, differences have been reported by Kaufman *et al* (1981) and ES. In this paper differences in the treatment of the Ising model will be reported. (It should be noted that results in this section are given in the context of approximation on regular lattices. The situation of external fields in the context of the exactly solved hierarchical lattices is discussed in § 6.)

A constraint exists on the scaling of fields conjugate to the order parameter (symmetry-breaking fields) at the high-coupling (discontinuity) fixed points. Nienhuis and Nauenberg (1975) argued that eigenvalues of these fields $\lambda_h$ should satisfy $\lambda_h = b^d$ at such points. From (17) and the results given in appendix 2, after some algebra one finds

$$\lambda_h = q\left(MB\exp(2BJ)\frac{1-\tanh^A(BJ)}{1+\tanh^A(BJ)}\right) + (1-q)\left(1+\tfrac{1}{2}zM(\exp(2BJ)-1)\frac{1-\tanh^{A-1}(BJ)}{1+\tanh^A(BJ)}\right). \tag{18}$$

The ferromagnetic discontinuity fixed point is found in the limit $J \to \infty$. From (18) one finds

$$\lambda_h = (1-q)[1+\tfrac{1}{2}zM(A-1)]+qMBA. \tag{19}$$

On substitution of the conservation conditions (7*a*) and (7*b*), (19) gives the correct scaling eigenvalue independent of $q$.

In appendix 2 general recursion relations for the Hamiltonian (17) are found. The coefficients specified by (iii) are used ($A = b$, $M = (2/z)(b^d - 1)/(b-1)$, $B = (b^{d-1}/M)$) in actual calculation. For the case $b = 3$ the relations are

$$J' = \tfrac{1}{4}M\ln(g_1g_2/g_3g_4) \tag{20}$$

$$h' = (1-q)(1-\tfrac{1}{2}zM)h + \tfrac{1}{4}zM\ln(g_1/g_2) \tag{21}$$

$$f' = (1-q)(1-\tfrac{1}{2}zM)f + \tfrac{1}{4}zM\ln(g_3/g_4) \tag{22}$$

where

$$g_1 = r^2s^3 + r^{-1}[s(w^{-2}+w^2)+s^{-1}]$$

$$g_2 = r^3s^{-3} + r^{-1}[s^{-1}(w^{-2}+w^2)+s]$$

$$g_3 = r^{-3}w^3 + r[w(s^{-2}+s^2)+w^{-1}]$$

$$g_4 = r^{-3}w^{-3} + r[w^{-1}(s^{-2}+s^2)+w]$$

and

$$r = \exp(BJ) \qquad s = \exp[(1-q+2Bq/z)h] \qquad w = \exp[(1-q+2Bq/z)f].$$

Each parameter $(J, h, f)$ and each pair of parameters forms an invariant subspace under the relations. In addition the following symmetries are evident: $(J, h, f) \rightarrow (-J, f, h)$ and changing the sign of the fields. To find some of these symmetries it is necessary to preserve both sublattices by choosing an odd-integer scale change. However, it is useful to continue the relations for arbitrary $b$ such that all the symmetries above are also continued. In appendix 2 a few tricks are employed to this end. It is noted that the symmetries of the relations are found for all $z$; therefore the approximation does not exhibit the absence of antiferromagnetic transitions in zero field on some close-packed lattices such as the triangular lattice. Restriction to open lattices has already been implicit in the assumption above of two sublattices. (The field $f$ cannot be defined on the close-packed lattices.)

Other authors have reported results for the antiferromagnetic region $(-J, h, 0)$ using a differential Migdal–Kadanoff approximation (continuation being made as described in appendix 2). Nagai and Toyonaga (1981) used $q = 0$ whereas Coniglio *et al* (1981) used $q = 1$. Both these works report a critical surface in agreement with other (non-renormalisation) studies (Binder and Landau 1980, Müller-Hartmann and Zittartz 1977). However, the renormalisation flows and fixed-point structures differ between $q = 0$ and $q = 1$. As described above the $(-J, h, 0)$ subspace, flows, fixed points and exponents can all be mapped directly onto the $(J, 0, f)$ subspace. Hence, for example, the result (19) for the $h$ field at the ferromagnetic discontinuity point implies directly the same result for the $f$ field at the antiferromagnetic discontinuity point. These subspaces contain all the fixed points and are discussed below in a joint fashion.

Figure 11 shows the flow found from (20), (21) and (22) in the case $q = 1$, $d = 2$ and $z = 4$. The critical fixed point on the $X$ axis (labelled C1) is unstable with respect to a second fixed point (labelled C2) on the critical surface. Coniglio *et al* (1981) also found this second fixed point. As will be reported below exponents at C2 differ from those at C1 which indicates a breakdown in universality between transitions with field (C2) and without field (C1). This breakdown is not expected of the exact model. The appearance of such 'satellite' fixed points as C2 has been noticed with Migdal–Kadanoff approximations to other models (Kaufman *et al* 1981). For $q < 1$ the relations develop a serious pathology. For each $q < 1$ there exist regions within which the relations exchange the sign of the fields. Figure 12 shows these regions. For $q < 1$ a critical surface exists in the 'good' region of the space. As $q$ is decreased C2 moves rapidly towards C1, merging with C1 (in this case $b = 3$ and $d = 2$) at $q = 0.944$ at which the scale dimension along the surface C1/C2 is marginal. Figure 12 also shows the critical surface and flow for $q = 0.94$; the universality breakdown described earlier in the case $q = 1$ has been lost. As $J \rightarrow \infty$ the critical surface for $q = 1$ is $h_c = zJ_c$, the correct result; for $q < 1$ the correct limit is only found as $b \rightarrow 1$. Continuing the relations for arbitrary $b$ as described in appendix 2 reveals that the pathological regions shrink and vanish in the limit $b \rightarrow 1$ and the differential transformation is assumed to show no such pathologies for all $q$. The qualitative form of the flows in figures 11 and 12 is found respectively for $q = 1$ and 0 as $b \rightarrow 1$. The $q = 1$ result presented by Coniglio *et al* (1981) is recovered; however, the $q = 0$ result presented by Nagai and Toyonaga (1981) is not found. Their result has the point C1 unstable with respect to the critical surface whereas this author finds (as in figure 12) that C1 is stable. This topology of

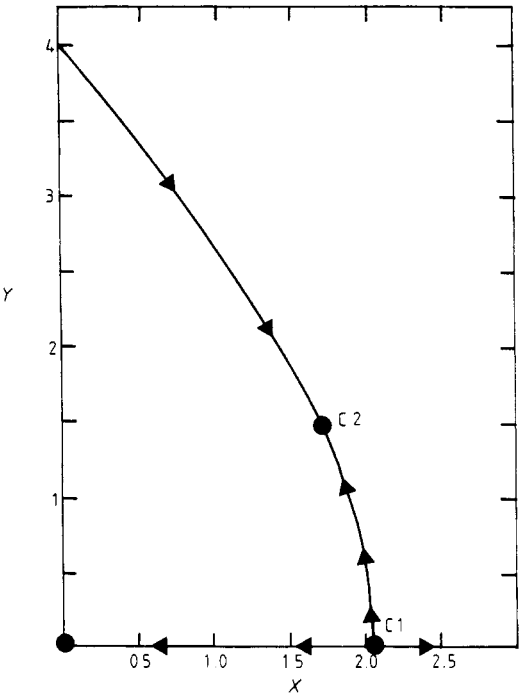


**Figure 11.** Flow in the $(-J, h, 0)$ and $(J, 0, f)$ subspaces in the case $b = 3$, $q = 1$, $d = 2$ and $z = 4$. The axes are $X = 1/|J|$ and $Y = h/|J|$ for $J < 0$ and $Y = f/|J|$ for $J > 0$. Fixed points are at C2, $X = 1.7249$ and $Y = 1.4854$, and C1, $X = 2.0781$ and $Y = 0$.

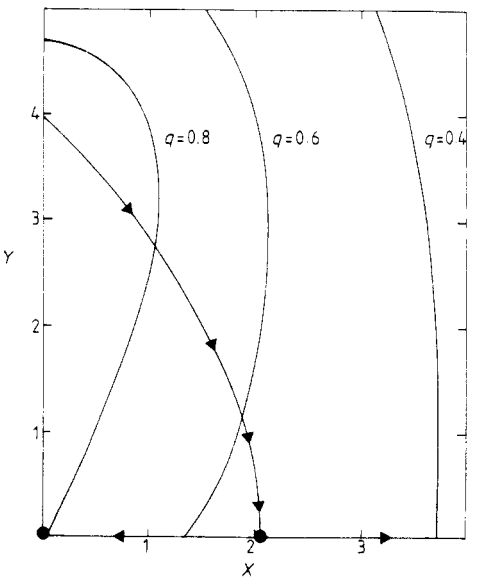


**Figure 12.** Pathological regions in the case $b = 3$, $d = 2$ and $z = 4$, and $q = 0.8$, 0.6 and 0.4. In the region bounded to the left of these lines, $h' < 0$ if $h > 0$ and *vice versa*. Also shown is the critical surface in the case $q = 0.94$. $X$ and $Y$ are defined as in figure 11.

a single fixed point on the $J$ axis, stable with respect to the critical surface, is that expected for the model. For $d > 1.9031$ the qualitative nature of the picture described above does not alter. At $d = 1.9031$ and $q = 1$ C2 merges with C1 and only C1 exists for all $q$ and $d < 1.9031$. As $d \to 1$ the transformation is exact, independent of $q$, and all critical surfaces are shrunk to the high-coupling fixed point. Table 2 gives the positions of fixed points for the case $z = 2d$ and $b = 1.1$.

**Table 2.** Fixed points C1 ($q = 1$ and 0, $z = 4$) and C2 ($q = 1$ and $z = 4$) and scaling dimensions ($b = 1.1$). $X = 1/|J|$ and $Y = h/|J|$ for $J < 0$ and $Y = f/|J|$ for $J > 0$.

| Fixed point C2 $d$ | $X^*$ | $Y^*$ | $Y_T$ | $Y_2$ | $Y_0$ |
|---|---|---|---|---|---|
| 2 | 1.9018 | 1.4929 | 0.8346 | −0.2291 | 1.9043 |
| 3 | 1.3910 | 4.9858 | 1.8797 | −2.9686 | 2.8872 |
| 4 | 1.1892 | 7.2657 | 2.8835 | −6.3224 | 3.8796 |
| **Fixed point C1** $d$ | $X^*$ | $Y^*$ | $Y_T$ | $Y_2(^{q=1}_{q=0})$ | $Y_0(^{q=1}_{q=0})$ |
| 2 | 2.2676 | 0 | 0.7534 | 0.1187<br>−1.9448 | 1.8814<br>1.7723 |
| 3 | 7.1419 | 0 | 0.9480 | 1.4408<br>−1.9778 | 2.5592<br>1.7473 |
| 4 | 19.90 | 0 | 0.9899 | 2.6992<br>−1.4754 | 3.3008<br>1.3286 |

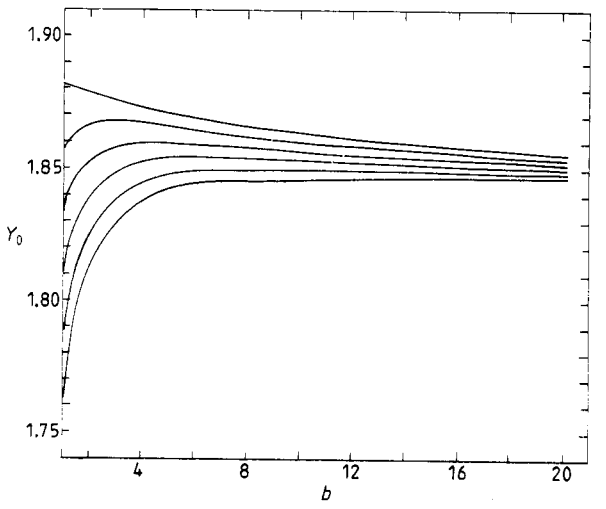


**Figure 13.** $Y_0$ against $b$ for the case $d = 2$ and $q = 0$, 0.2, 0.4, 0.6, 0.8 and 1.0 (from the bottom to the top curve, respectively). As $b \to 1$ values are respectively 1.7627, 1.7864, 1.8102, 1.8339, 1.8576 and 1.8813.

Considering now the results for exponents, universality with respect to $z$ is found for all exponents. C1 and C2 are unstable with respect to $f$ in the $(-J, h, 0)$ space and to $h$ in the $(J, 0, f)$ space; the associated scale dimension is denoted $Y_0$. Exponents depend on $b$ and $d$ and in addition $Y_0$ and $Y_T$ at C2 depend on $q$. The thermal scaling dimension $Y_T$ at C1 has already been described in figure 6. Figure 13 shows the dependence of $Y_0$ on $b$ and $q$ at C1 in the case $d = 2$. Similar curves are found in higher dimensions. For large $b$ the dependence on $q$ disappears, whereas for small $b$ a marked dependence is shown. In the limit $b \to 1$ the values of $q$ which give the known results are close to one (that $q = 1$ gives good values has been noticed before). Respectively for $d = 2$, 3 and 4, $q = 0.946$, 0.916 and 0.857 gives $Y_0 = 1.875$, 2.485 (Le Guillou and Zinn Justin 1980, see also Roskies 1981) and 3. Table 2 presents exponents at C1 and C2 for $q = 1$ and $q = 0$ and $b = 1.1$. The results at C2 are as different from the known results as those at C1 are. It is noted that whereas at C1 the specific heat exponent $\alpha$ is bounded, $\alpha < 0$ (a result found on all hierarchies (Melrose 1983)). At C2 this is not so; with $b = 1.1$ for $d = 3$ and 4 one finds respectively $\alpha = 0.4042$ and 0.6128 ($\alpha$ is positive for all $d > 2.2738$).

## 6. The Migdal–Kadanoff hierarchies

Berker and Ostlund (1979) observed that the string $(ABM)$ decorations could be used to define a *hierarchical lattice*. The hierarchies are defined by an iterative generation. Starting from a single bond at each step, every bond is decorated with the chosen decoration. For the case $A = 2$, $M = 2$ and $B = 1$ the first two iterative steps are shown in figure 14. The family of hierarchies based on the string decorations is referred to below as the MKH. The iteration may be reversed by decimating all spins

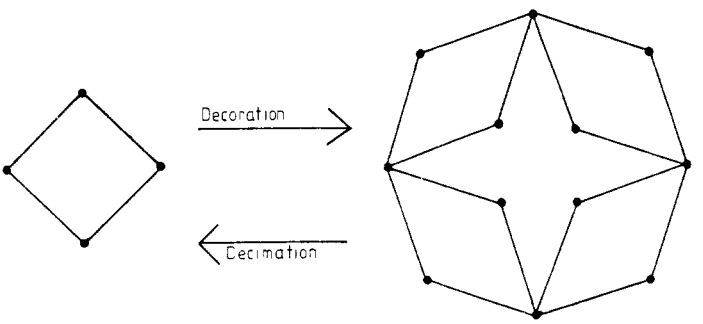


**Figure 14.** First two steps in the generation of the hierarchy with $A = 2$, $M = 2$ and $B = 1$.

at the lowest level resulting in an exact renormalisation transformation. Recursion relations found under Migdal–Kadanoff approximations on regular lattices constitute exact solutions of models on the MKH.

Many examples of hierarchies exist (Kaufman and Griffiths 1981, Melrose 1983). Other decimation approximations on regular lattices (Sneddon and Barber 1977, Reynolds *et al* 1977, see also Chao 1981) can also be considered as exact on a hierarchy. Understanding of these lattices is in its early stages. Kaufman and Griffiths

(1981) have reported results including the existence of the thermodynamic limit and the relationships between features of the lattices and the nature of phase transitions thereon. The lattices lack translational invariance and are highly inhomogeneous, yet it is clear that they can support a wide variety of phase transitions and critical phenomena. However, the lattices possess unusual features not found on regular lattices. Kaufman and Griffiths (1982) argue that the susceptibility is infinite throughout the high-temperature region. Other fractal lattices on which decimation renormalisation groups can be found exactly (Gefen *et al* 1980, Dhar 1977, Nelson and Fisher 1975) do not show phase transitions above zero temperature.

In this section several features of the MKH are reported. Firstly the $M$ and $B$ coefficients enter the recursion relations in a simple fashion (cf equation (A1) in the case $r = 1$). The temperature may be rescaled in the relations such that the product $BM$ is constant, leaving exponents and phase diagram topology unchanged. It is noted that the coefficients (1$b$), (1$c$) and (4) all have the same product $BM$ (due to bond conservation) and hence are all equivalent under temperature rescaling. Further the universality in $z$ shown by the coefficients (iii) in equation (11) results, because the product $BM$ is independent of $z$.

The hierarchical lattices are examples of fractal lattices (Mandelbrot 1977). On these lattices definitions of dimension lead to non-integer values. Whereas on regular lattices different definitions of dimension agree, on fractal lattices they usually do not. Several authors (Mandelbrot 1977, Dhar 1977, McKenzie 1981) have introduced dimension definitions. The fractal dimension, $D$ (Mandelbrot 1977), characterises how the number of bonds on the lattice grows with its size. It is necessary (Kaufman and Griffiths 1981) to introduce a definition of length on the lattices. This is set by defining the scale change, $b$, associated with the renormalisation (decimation) step. Taking each bond of length unity and following McKenzie (1981), the distance between two vertices is set as the number of bonds on the shortest path on the lattice between the vertices. On the MKH one finds $b = A$ and as each decoration contains $ABM$ bonds

$$D = \ln ABM / \ln A. \tag{23}$$

It is noted that for the approximation coefficients $D = d$ the dimension of the regular lattice under consideration. Following directly the definition of Dhar (1977), in appendix 3 it is found that this also leads to (23). Another parameter of interest is the connectivity, $Q$ (Gefen *et al* 1980). This characterises how the number of bonds which need be cut to separate a finite hierarchy from a larger unit grows with its size. One finds (Melrose 1983)

$$Q = \ln MB / \ln A. \tag{24}$$

$D$ (or $d$) $= 1 + Q$ on regular lattices, as found on the MKH, but in general $D > 1 + Q$ on the hierarchies (Melrose 1983). In contrast to most other fractal lattices different definitions of dimension agree on the MKH.

A duality relationship between members of the MKH was discussed earlier in § 2. From (23) it is seen that, in general, dual MKH do not have the same dimension (a notable exception to this being the case $M$ (or $B$) $= A$ and $D = 2$).

The treatment of external fields on the MKH was discussed by Yeomans and Fisher (1981). Only the choice $q = 1$ leads to a Hamiltonian whose form is invariant under renormalisation (in contrast to the approximations where a range of $q$ was available). The fields are assigned to the spins in proportion to their coordination on the lattice.

This field assignment was adopted in the computation of results presented below. It is noted that the breakdown in universality between transitions in and out of fields is an exact property of the MKH.

The MKH exhibit a complete breakdown in universality. Under the temperature rescaling one may set either $M = 1$ or $B = 1$ so the MKH can be parametrised by just two parameters and here $M$ and $A$ are chosen. Figure 15 shows contours of constant $Y_T$ and $Y_0$ found at the fixed points C1 in the $(M, A)$ space. (The formal difficulties

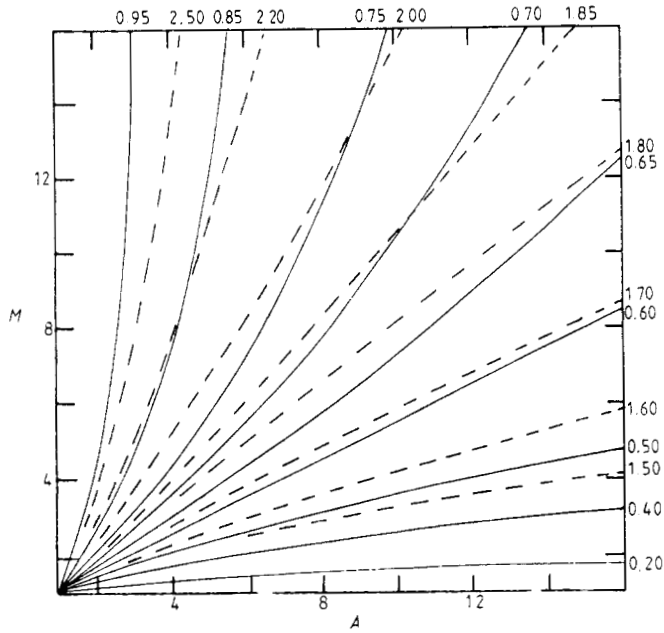


**Figure 15.** Contours of constant $Y_T$ (full curves) and $Y_0$ (broken curves) in the $(M, A)$ space. Values of each contour given on the side of the figure.

of interpreting these exponents on the hierarchies are noted (Kaufman and Griffiths 1981, Dhar 1977); they are, however, the implicit exponents from which the well defined thermodynamic exponents may be found.) The contours of $Y_T$ and $Y_0$ are not the same and do not follow the curves of constant $D$. Similar results are found for the thermodynamic exponents, each exponent having distinct contours. At the fixed points C2 the same picture emerges and, furthermore, the contours at C2 do not correspond with the contours of the same exponent at C1.

## 7. Conclusions

The bond-moving approximation was shown to allow a wide variation in the formation of string-decorated superlattices. On the hypercubic lattices an optimal approximation was found. This gave a maximal free energy estimate over the full coupling range. The present ambiguity in the literature over which approximation to use is removed (although this is a result of small significance as all the varieties in use give the same

exponents and phase diagram topology). On extending the bond-moving to all regular lattices it was found that, whilst on most lattices strings of length $b$ are formed, on some lattices strings of other lengths are found. Universality is broken by the bond-moving. An *ad hoc* generalisation was undertaken by directly investigating the dependence of results on the string length. An approximation continuous in both dimension and coordination was found to show both universality and qualitatively correct critical couplings. Another approximation was found to give a differential transformation which possessed the correct duality of planar lattices. The phase diagram of the Ising model with external fields was examined. The ambiguity in the treatment of external fields was investigated directly with a free parameter $q$. A diagram in agreement with that expected was found only for the differential transformation with $q = 0$. For $b > 1$ the $q < 1$ recursion relations were found to be pathological. The $q = 1$ transformations broke an expected symmetry between transitions in and out of fields (notably for both ferromagnetic and antiferromagnetic transitions). Finally the recursion relations were discussed in the context of exact solutions on hierarchical lattices, the MKH. Unlike other hierarchical and fractal lattices different definitions of dimension agree on the MKH. Duality transformations were found to relate lattices of different dimension. The MKH were found to exhibit a complete breakdown in the concept of universality.

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## Appendix 1. Optimisation of the string decoration

Without loss of generality the case of a model with a single coupling parameter $J$ is considered. For a decoration involving $r$ types of strings specified by coefficients $(M_i, B_i)$ the recursion relations assume the form

$$J' = \sum_{i=1}^{r} M_i R_A(B_i J) \tag{A1.1}$$

$$K = \sum_{i=1}^{r} M_i k_A(B_i J) \tag{A1.2}$$

where $R_A(x)$ and $k_A(x)$ are some functions which depend on the lengths of the strings $A$. (A1.1) for $r = 1$ should be compared with the equivalent equation (12).

Firstly the general problem is discussed. The free energy is found from iteration of (A1.1) and summation of contributions (A1.2). One finds (Nauenberg and Nienhuis 1974)

$$-\beta f = \frac{1}{b^d} K(J_1, P_1) + \frac{1}{b^{2d}} K(J_2, P_2) + \frac{1}{b^{3d}} K(J_3, P_3) + \ldots \tag{A1.3}$$

where $-\beta f$ is the free energy per bond, $J_1$, $J_2$, etc, are the couplings at the first, second, etc, iterations and $P_1$, $P_2$, etc, are some vectors of variational parameters $p_{ij}$ which

initially are allowed to vary with iteration step $i$. Optimisation of (A1.3) with respect to $p_{ij}$ is given by

$$\frac{\mathrm{d}(-\beta f)}{\mathrm{d}p_{ij}} = \frac{1}{b^{id}} \frac{\partial K(J_i, P_i)}{\partial p_{ij}} + \frac{1}{b^{(i+1)d}} \frac{\partial K(J_{i+1}, P_{i+1})}{\partial J_{i+1}} \frac{\partial J_{i+1}}{\partial p_{ij}} + \frac{1}{b^{(i+2)d}} \frac{\partial K(J_{i+2}, P_{i+2})}{\partial J_{i+2}} \frac{\partial J_{i+2}}{\partial J_{i+1}} \frac{\partial J_{i+1}}{\partial p_{ij}} + \ldots = 0. \tag{A1.4}$$

The following solution is found:

$$\frac{\partial K(J_i, P_i)}{\partial p_{ij}} = 0 \qquad \text{and} \qquad \frac{\partial J_{i+1}}{\partial p_{ij}} = 0. \tag{A1.5}$$

If (A1.5) is found for all $j$ independent of the coupling then optimisation of the full series follows and, trivially, parameters will not vary with the iteration step $i$.

In the case of concern here (A1.1) and (A1.2) have the same form so proof of one condition of (A1.5) implies the other. Note that the derivatives of (A1.1) have a similar form which it is straightforward to optimise, as below, leading to the same solution. The coefficients $B_i$ and $M_i$ are constrained by the conservation conditions (3$a$) and (3$b$). $M_1$ and $B_1$ are treated as dependent parameters and the rest are varied. From (3$a$) and (3$b$)

$$M_1 = X - \sum_{i=2}^{r} M_i \tag{A1.6}$$

$$B_1 = \frac{1}{M_1}\left(Y - \sum_{i=2}^{r} M_i B_i\right) \tag{A1.7}$$

where $X$ and $Y$ are constants. Now $K$ is rewritten

$$K = M_1(M_i) k_A(B_1(M_i B_i) J) + \sum_{i=2}^{r} M_i k_A(B_i J). \tag{A1.8}$$

Directly one finds

$$\partial K/\partial B_j = J M_j(-k'_A(B_1 J) + k'_A(B_j J)), \tag{A1.9}$$

where $K'_A(x)$ denotes $\mathrm{d}K_A(x)/\mathrm{d}x$; one also finds

$$\partial K/\partial M_j = J M_1 k'_A(B_1 J)(\partial B_1/\partial M_j) - k_A(B_1 J) + k_A(B_j J) \tag{A1.10}$$

$$\frac{\partial B_1}{\partial M_j} = \frac{1}{M_1^2}\left(Y - B_j M_1 - \sum_{i=2}^{r} B_i M_i\right). \tag{A1.11}$$

On substitution of $B_j = B_1$ for all $j$ the above derivatives vanish. The condition (A1.5) is satisfied and optimisation follows. The optimal decoration is the homogeneous one with all strings the same.

## Appendix 2. The recursion relations

In this appendix recursion relations are found for the Hamiltonian (17). Decimation of spins along a string is best carried out as product of transfer matrices, as expressed in equation (12). The $B$ and $M$ coefficients enter the relations in a simple fashion.

Site terms on the decorations may be included in the transfer matrices. However, the superlattice spins at the end of the strings have a different coordination from those on the decorations. To avoid having to deal with different matrices at the ends of the strings on including the site terms, a factor $\frac{1}{2}zM(1-q)$ of the fields is added and subtracted at the superlattice sites. The resulting transfer matrix for a bond of a string is

$$T = \begin{pmatrix} rs & r^{-1}w \\ r^{-1}w^{-1} & rs^{-1} \end{pmatrix} \tag{A2.1}$$

where

$$r = \exp(BJ) \qquad s = \exp[(1-q+2Bq/z)h] \qquad w = \exp[(1-q+2Bq/z)f].$$

The staggered field $f$ alternates in sign along the strings; as a result the transfer matrix is transposed at every other bond along a string. The matrix product is of the form

$$(TT^{\mathrm{T}}TT^{\mathrm{T}}T \ldots T^{\mathrm{T}}T) \tag{A2.2}$$

where $T^{\mathrm{T}}$ denotes the transposed matrix. In (A2.2) $A$ has been assumed to be odd. It is desirable to continue the product for arbitrary $A$ such that all the symmetries found at odd scale change (see § 5) are preserved. This condition is imposed by the tricks below.

(i) Following Nagai and Toyonaga (1981) and Coniglio *et al* (1981), non-integer powers of negative eigenvalues of $T$ are handled by the replacement

$$\lambda^x = \lambda|\lambda|^{x-1}. \tag{A2.3}$$

(ii) The matrix product (A2.2) is written

$$(TT^{\mathrm{T}})^{(A-1)/2}T. \tag{A2.4}$$

The first product in (A2.4) is carried out via diagonalisation of the matrix $(TT^{\mathrm{T}})$. After some algebra and solving for the new parameters $J'$, $h'$ and $f'$, the following relations are found:

$$J' = \tfrac{1}{4}M \ln(g_1g_2/g_3g_4)$$

$$h' = (1-q)(1-\tfrac{1}{2}zM)h + \tfrac{1}{4}zM \ln(g_1/g_2)$$

$$f' = (1-q)(1-\tfrac{1}{2}zM)f + \tfrac{1}{4}zM \ln(g_3/g_4)$$

where

$$g_1 = rsp_1 + r^{-1}w^{-1}p_2 \qquad g_2 = r^{-1}wp_1 + rs^{-1}p_2$$

$$g_3 = rsp_2 + r^{-1}w^{-1}p_3 \qquad g_4 = r^{-1}wp_2 + rs^{-1}p_3$$

and $p_1$, $p_2$ and $p_3$ are the elements of $(TT^{\mathrm{T}})^{(A-1)/2}$:

$$p_1 = (sw^{-1}+ws^{-1})^2\lambda_1^{(A-2)/2} + (\lambda_1 - r^2s^2 - r^{-2}w^2)^2\lambda_2^{(A-1)/2}$$

$$p_2 = (sw^{-1}+ws^{-1})(\lambda_1 - r^2s^2 - r^{-2}w^2)(\lambda_1^{(A-1)/2} - \lambda_2^{(A-1)/2})$$

$$p_3 = (sw^{-1}+ws^{-1})^2\lambda_2^{(A-1)/2} + (\lambda_1 - r^2s^2 - r^{-2}w^2)^2\lambda_1^{(A-1)/2}$$

and

$$\lambda_{1,2} = \tfrac{1}{2}[\![r^2(s^2+s^{-2}) + r^{-2}(w^2+w^{-2}) \pm \{[r^2(s^2-s^{-2}) + r^{-2}(w^2-w^{-2})]^2 + 4(sw^{-1}+ws^{-1})^2\}^{1/2}]\!].$$

## Appendix 3. Dhar's definition of dimension

Dhar (1977) defined the dimension of a lattice by the known properties of the low-temperature behaviour of the spherical model. On the hierarchies the Hamiltonian of this model is written

$$-\beta H = \sum_{ij} [-(\xi_i - \xi_j)^2 - \lambda(\xi_i^2 + \xi_j^2)] \tag{A3.1}$$

where the $\xi_i$ represent continuous spins $-\infty < \xi_i < \infty$ and the reduced coupling is taken as unity. On the hierarchies the spherical constraint on the spins (the second term in (A3.1)) is applied in proportion to their coordinations in an analogous fashion to the application of an external field. It is straightforward to carry out a decimation transformation by integration. For the case $A = 2$, $B = 1$ and arbitrary $M$ one finds the new Hamiltonian

$$-\beta H = \sum_{ij} -[M/2(\lambda+1)](\xi_i - \xi_j)^2 - [M(\lambda^2 + 2\lambda)/(\lambda+1)](\xi_i^2 + \xi_j^2). \tag{A3.2}$$

The spins are now rescaled (in proportion to their coordinations) such that the new coupling is again unity. A new constraint parameter

$$\lambda' = 2(\lambda^2 + 2\lambda) \tag{A3.3}$$

is found; $\lambda = 0$ is a fixed point of this relation. The renormalisation transformation of the free energy per bond is

$$f(\lambda) = (2M - 1)\ln[M/(4\lambda + 4)]/M + (1/2M) f(2\lambda^2 + 4\lambda). \tag{A3.4}$$

Now following Dhar (1977), for small $\lambda$, $f$ is assumed to have the form

$$f(\lambda) = f(0) + A\lambda^{D/2} + \text{terms of higher order} \tag{A3.5}$$

which defines $D$, the dimensionality parameter. From (A3.4) and (A3.5) one finds

$$D = \ln 2M / \ln 2. \tag{A3.6}$$

This is just the result (23), the fractal dimension; in this case $A = 2$, $B = 1$. It is straightforward to introduce general $B$ and $A$ (when $A$ is even simply iterate (A3.3); when $A$ is odd it can be treated from the case $A = 3$).

*Note added in proof.* In addition to § 5: When $d = 1.9031$ and the fixed points merge at $q = 1$, the curves $Y_T(d)$ and $Y_0(d)$ at C2 merge with discontinuous slope onto those at C1. The curvature of the critical surface for small $Y$ is found as $b \to 1$ and respectively for $q = 1$ and $q = 0$ some 100% and 4% greater than that of the MHZ solution.

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